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Classification of the Five-Dimensional Lie Superalgebras Over the Real Numbers

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The purpose of this contribution, is to initiate a classification of Lie superalgebras (LS) of dimension five, over the base field $\mathbb{R}$ of real numbers. We use the "*graded skew-symmetry*" and the "*graded Jacobi identity*" in order to get restrictions for the commutators and anticommutators of an arbitrary five-dimensional Lie superalgebra $L = L_0 \oplus L_1$.

KEY WORDS: Lie superalgebra.

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1. INTRODUCTION

About thirty years ago, there appeared in physics (see Neveu and Schwarz, 1971; Wess and Zumino, 1974; Salam and Strathdee, 1974) the new symmetry principle called *supersymmetry*. The significant feature of this symmetry is that it allows bosons and fermions, having different spin parities, to be considered within the same multiplets. The generators of supersymmetry transformations form a mathematical structure, known as Lie superalgebra. The theory of LS has developed remarkably quickly, both in physics and in mathematics. From the mathematical point of view, the major effort has been directed towards establishing a theory of simple LS (see Pais and Rittenberg, 1975; Freund and Kaplansky, 1976) and their representations (see Corwin *et al.*, 1975; Scheunert *et al.*, 1977; Kac, 1978). A classification of the former, was made by Kac (see Kac, 1977) in the finite-dimensional case as the first major step in the subject. Representation theory is very much more difficult for superalgebras than it is for ordinary Lie algebras (see Balantekin and Bars, 1982, 1984; Berele and Regev, 1987; Palev, 1989; Van der Jeugt, 1987). The simple LS are those which are commonly used to describe internal symmetries of elementary particles and so naturally there has been an emphasis

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in the literature on the simple algebras. A notable exception to this, has been the use of nilpotent Heisenberg superalgebras as supersymmetry algebras for the harmonic oscillator (see Beckers *et al.*, 1988a,b). However, the simple LS form a very small subset of all possible LS, yet the non simples have received very little attention even within mathematics. As a first step towards providing material for a study of more general LS, there exists a classification of all real LS, which are not Lie algebras, up to dimension four (see Backhouse, 1978a), a classification of all nilpotent LS up to dimension five (see Hegazi, 1999a,b) and a classification of all complex LS of dimension five (see Matiadou and Fellouris, 2005). In the present paper, we extend the scope of the classification of LS to dimension five, over the real numbers.

We recall that, a Lie superalgebra $L = L_0 \oplus L_1$ is a superalgebra over a base field of characteristic different from 2, with a bilinear bracket operation [. , .] satisfying the following axioms:

1. $[x, y] = -(-1)^{\alpha\beta}[y, x]$ (*graded skew-symmetry*)
2. $(-1)^{\alpha\gamma}[[x, y], z] + (-1)^{\alpha\beta}[[y, z], x] + (-1)^{\beta\gamma}[[z, x], y] = 0$ (*graded Jacobi identity*)

for all $x \in L_\alpha, y \in L_\beta, z \in L_\gamma, \alpha, \beta, \gamma \in Z_2$.

Let $L = L_0 \oplus L_1$ be a Lie superalgebra. L_0 is called the even part of L and it is a Lie algebra. L_1 is called the odd part of L and it is a L_0-module.

Two LS $L = L_0 \oplus L_1$ and $L' = L'_0 \oplus L'_1$ are said to be equivalent, if there exists a graded isomorphism f, which preserves the bracket product and is homogeneous of degree zero, i.e. $f(L_0) = L'_0$ and $f(L_1) = L'_1$. It is often convenient to think of f, as being determined by its restrictions to the even and odd parts.

We can ask the question: given a Lie algebra L_0 and an L_0-module M, how many inequivalent Lie superalgebras $L = L_0 \oplus L_1$ can we construct, where L_1 and M are equivalent as L_0-modules. Answering this question in the fifth dimension case, is the basis for our classification. It is very convenient, both for the carrying out of our computations and for the tabular representation of our results, to distinguish the following two types of Lie superalgebras. A Lie superalgebra L is trivial, if $[L_1, L_1] = \{0\}$; otherwise, L is a non trivial one.

A secondary distinction on the set of LS of a given dimension, is according to the dimensions of the even and odd parts. Thus, L is a (m, n)-Lie superalgebra and has dimension $m + n$, if $\dim L_0 = m$ and $\dim L_1 = n$

Remark 1.1. From "*graded skew-symmetry*" and "*graded Jacobi identity*", we obtain the following relations, which hold for all $a, b, c \in L_0$ and $\alpha, \beta, \gamma \in L_1$:

$$[a, b] = -[b, a] \tag{1.1}$$

$$[a, \alpha] = -[\alpha, a] \tag{1.2}$$

$$[\alpha, \beta] = [\beta, \alpha] \tag{1.3}$$

$$[[a, b], c] + [[b, c], a] + [[c, a], b] = 0 \tag{1.4}$$

$$[[a, b], \alpha] + [[b, \alpha], a] + [[\alpha, a], b] = 0 \tag{1.5}$$

$$[[a, \alpha], \beta] + [[\alpha, \beta], a] - [[\beta, a], \alpha] = 0 \tag{1.6}$$

$$[[\alpha, \beta], \gamma] + [[\beta, \gamma], \alpha] + [[\gamma, \alpha], \beta] = 0 \tag{1.7}$$

2. FIVE-DIMENSIONAL LIE SUPERALGEBRAS

Coming now to the mechanics of our classification of all five-dimensional real LS, we begin by considering all trivial real LS for each of the possible dimension types (m, n), with $m + n = 5$. We require, as basic input, a list of all inequivalent real Lie algebras of dimension m, for $m = 1, 2, 3, 4$. Our source is the work of Mubarakzyanov (see Mubarakzyanov, 1963; Patera *et al.*, 1976). For each such algebra, we must determine its inequivalent n-dimensional representations, for $n = 5 - m$. In each case, the Lie algebra and its representation form a trivial LS of dimension five. The decomposable trivial algebras, although not included in the tabulations, are important in that they are needed as building blocks for the non trivial algebras. In fact, any non trivial algebra can be trivialized simply by putting to zero all anti-commutators. The trivial algebra so obtained, can be decomposable even if the starting algebra is not. Non trivial algebras are obtained by reversing the trivialization process.

2.1. (4,1)-Dimensional Lie Superalgebras

Let $L = L_0 \oplus L_1$ be a (4, 1)-LS and $\{a_1, a_2, a_3, a_4; \alpha\}$ a set of generators. In particular, a_1, a_2, a_3, a_4 are the generators of the even part L_0 and α is the generator of the odd part L_1. The one-dimensional representation $\rho : L_0 \to L_1$ is defined by:

$$\rho(a_i)(\alpha) \equiv [a_i, \alpha] = \lambda_i \alpha, \quad \text{which holds for all} \quad i = 1, 2, 3, 4.$$

Then, we either have one of the following two cases:

1. $\lambda_i = 0$, for all $i = 1, 2, 3, 4$.
2. There is at least one $\lambda_i \neq 0$, for example $\lambda_4 \neq 0$.

2.1.1. Trivial (4,1)-Dimensional Lie Superalgebras

In case 1, any (4, 1)-trivial LS is an ordinary Lie algebra of dimension four (see Patera *et al.*, 1976).

In case 2, it is very convenient to change the basis in L_0. So, we define the new basis $\{a'_1, a'_2, a'_3, a'_4\}$, as:

$$a'_i = a_i - \frac{\lambda_i}{\lambda_4} a_4, \quad i = 1, 2, 3$$
$$a'_4 = a_4,$$

Now, we have:

$$[a_i', \alpha] = 0, \quad \text{for} \quad i = 1, 2, 3$$

$$[a_4', \alpha] = \lambda_4 \alpha, \quad \lambda_4 \neq 0.$$

In the sequel, it is easier to use a_i instead of a_i' for the labelling of the new basis elements.

Remark 2.1.

1. The scalar λ_4 can be reduced to unity by scaling a_4.
2. If we call h the kernel of ρ, then we observe that h is a three-dimensional ideal of L_0, generated by a_1, a_2, a_3, with a_4 acting on h as an external derivation. This means that the following relation holds for all $i, j = 1, 2, 3$, where $i \neq j$:

$$[a_4, [a_i, a_j]] = [[a_4, a_i], a_j] + [a_i, [a_4, a_j]]$$

3. The quotient Lie algebra $L_0/\ker(\rho)$ is isomorphic to the one-dimensional Abelian Lie algebra. Thus, we have:

$$L_0 = h + \mathbb{R}\, a_4.$$

Using all inequivalent forms of the three-dimensional real Lie algebra h and the *graded Jacobi identity*, we finally find 18 pairwise non isomorphic families of trivial (4,1)-dimensional LS plus 1, namely the LS E^{10}. The latter one does not depend on any parameters, but it is non isomorphic to any other LS from the above families.

2.1.2. Non Trivial (4,1)-Dimensional Lie Superalgebras

From the "*graded Jacobi identity*", we have:

$$[a_i, [\alpha, \alpha]] = 2[[a_i, \alpha], \alpha], \quad i = 1, 2, 3, 4 \tag{2.8}$$

$$[\alpha, [\alpha, \alpha]] = 0. \tag{2.9}$$

In case 1, L_0 is a 4-dimensional Lie algebra with $[L_0, L_1] = 0$. Using the relation (2.8), we get that $[\alpha, \alpha]$ belongs to the center of L_0.

In case 2, the relation (2.8) gives $[a_i, [\alpha, \alpha]] = 0$ for all $i = 1, 2, 3$. So, we conclude that $[\alpha, \alpha]$ belongs to the center of the ideal h.

This analysis leads to 7 families of non trivial (4,1)-dimensional LS, which are pairwise non isomorphic plus 9 non isomorphic LS, which are not depending on parameters.

2.2. (1,4)-Dimensional Lie Superalgebras

Let $L = L_0 \oplus L_1$ be a (1,4)-LS and $\{a; \alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ a set of generators.

2.2.1. *Trivial (1,4)-Dimensional Lie Superalgebras*

The action of a on L_1 is completely determined by a real 4×4 matrix. Thus, there exist 13 pairwise non isomorphic families of (1,4)-dimensional trivial Lie superalgebras plus 2 non isomorphic LS not depending on parameters, all arising from the rational canonical form of a real 4×4 matrix.

2.2.2. *Non Trivial (1,4)-Dimensional Lie Superalgebras*

Using the "*graded Jacobi identity*", we have:

$$[[\alpha_i, \alpha_i], \alpha_i] = 0, \quad i = 1, 2, 3, 4. \tag{2.10}$$

Let $[\alpha_i, \alpha_j] = S_{ij}\ a,\ i, j = 1, 2, 3, 4$ and $S_{ij} \in \mathbb{R}$. So, the matrix $S = (S_{ij})$ is a 4×4 real symmetric matrix and by linear transformations takes the form $S = \text{diag}\ (s_1, s_2, s_3, s_4)$.

Here, we have to examine the following three cases:

(1) If $s_i = 0$ for all $i = 1, 2, 3, 4$, then the LS is a trivial one.

(2) If there is at least one $s_i \neq 0$, let's say $[\alpha_1, \alpha_1] = s_1 a$, where $s_1 \neq 0$, then

$$[a, \alpha_1] = 0.$$

Using again the "*graded Jacobi identity*", we have:

$$[a, \alpha_i] = 0, \quad i = 2, 3, 4.$$

Therefore, when $s_1 s_2 s_3 = 0$, the LS decomposes. So, in order to obtain an indecomposable LS, we must have $s_i \neq 0$ for all i.

(3) If $s_i \neq 0$ for all $i = 1, 2, 3, 4$, then by scaling each α_i we can take $s_i = \pm 1$ for all i. So, we get three inequivalent forms of the diagonal matrix $S = \text{diag}(s_1, s_2, s_3, s_4)$, as follows:

$$\text{diag}(1, 1, 1, 1), \quad \text{diag}(1, 1, 1, -1), \quad \text{diag}(1, 1, -1, -1).$$

This analysis leads to 3 pairwise non isomorphic non trivial (1,4)-dimensional LS.

2.3. (3,2)-Dimensional Lie Superalgebras

Let $L = L_0 \oplus L_1$ be a (3,2)-LS and $\{a_1, a_2, a_3; \alpha_1, \alpha_2\}$ a set of generators.

2.3.1. Trivial (3,2)-Dimensional Lie Superalgebras

From the classification over $\mathbb{R}$ of the 3-dimensional Lie algebras, we can get 10 inequivalent cases for the Lie algebra L_0. We conclude the results by using all these different forms of L_0 and the "*graded Jacobi identity*," which holds for all $i, j = 1, 2, 3$ and $k = 1, 2$:

$$[[a_i, a_j], \alpha_k] + [[a_j, \alpha_k], a_i] + [[\alpha_k, a_i], a_j] = 0.$$

So, we finally find 25 pairwise non isomorphic families of trivial (3,2)-dimensional LS plus 8 more non isomorphic LS, which are not depending on any parameters.

2.3.2. Non Trivial (3,2)-Dimensional Lie Superalgebras

From the "*graded Jacobi identity*", we have:

$$[[a_i, \alpha_j], \alpha_k] + [[\alpha_j, \alpha_k], a_i] - [[\alpha_k, a_i], \alpha_j] = 0, \quad \text{for all}$$
$$i = 1, 2, 3, \quad j, k = 1, 2$$

and

$$[[\alpha_i, \alpha_j], \alpha_k] + [[\alpha_j, \alpha_k], \alpha_i] + [[\alpha_k, \alpha_i], \alpha_j] = 0, \quad \text{for all } i, j, k = 1, 2.$$

Using these two relations and the 33 non isomorphic trivial (3,2)-dimensional LS that we found before, we get 13 pairwise non isomorphic families of (3,2)-non trivial LS plus 18 LS not depending on parameters.

2.4. (2,3)-Dimensional Lie Superalgebras

Let $L = L_0 \oplus L_1$ be a (2,3)-LS and $\{a_1, a_2; \alpha_1, \alpha_2, \alpha_3\}$ a set of generators. Working as in the case of (3,2)-dimensional LS, we find 15 pairwise non isomorphic families of trivial (2,3)-dimensional LS plus 3 more LS, not depending on parameters. For the non trivial case, we find 24 pairwise non isomorphic families plus 27 LS, which are not depending on parameters. It is worthwhile paying some attention to a special case of this type, namely those denoted $(2A_{1,1} + 3A)^i$, for all $i = 1, 2, \ldots, 23$. These LS have the two-dimensional Abelian Lie algebra $2A_{1,1}$ as even part and the zero representation on the odd part. Now, imposing the "*graded Jacobi identity*" on any triple of generators, does not provide us with any restrictions on anticommutators. Hence, for an arbitrary anticommutator, we can write:

$$[\alpha_i, \alpha_j] = S_{ij}\ a + T_{ij}\ b,$$

where S_{ij}, T_{ij} are elements of the real 3×3 symmetric matrices S, T, respectively and $a = a_1$, $b = a_2$. For the sequel, we use the symbols α, β, γ, instead of α_1, α_2, α_3, respectively, in order to make our computations easier.

Thus, our problem is reduced to that of determining all possible inequivalent forms of a pair of real 3×3 symmetric matrices. In order to apply standard results (see Brebner and Grad, 1982; Bunse-Gerstner, 1984; Garvey *et al.*, 2003), we have to distinguish two cases:

I. at least one of the matrices S and T is non-singular,
II. both matrices S and T are singular.

In the first case, using transformation of the form:

$$\begin{pmatrix} \alpha' \\ \beta' \\ \gamma' \end{pmatrix} = P \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \tag{2.11}$$

on the set of odd generators, where P is a 3×3 real matrix and bearing in mind that we can permute, scale or change sign on the set of even generators a and b, we can reduce the pair S, T simultaneously into the forms $S' = P^T\ S\ P$ and $T' = P^T\ T\ P$, where:

(1) $S' = \mathrm{diag}(1, 1, 1)$, $\quad T' = \mathrm{diag}(1, q, r)$, $\quad 1 \geq |q| \geq |r| \geq 0$

(2) $S' = \mathrm{diag}(1, 1, -1)$, $\quad T' = \begin{pmatrix} p' & \kappa' & \lambda \\ \kappa' & q' & \mu \\ \lambda & \mu & r \end{pmatrix} = \begin{pmatrix} T_1 & T_2 \\ T_2^T & r\mathrm{I}_1 \end{pmatrix}.$

In case (1), we have the LS:

$$[\alpha, \alpha] = a + b, \quad [\beta, \beta] = a + qb, \quad [\gamma, \gamma] = a + rb,$$

which for $q = 1$ or $q = r$, is decomposable. For $q \neq 1$ and $q \neq r$, we use the invertible linear transformation:

$$\begin{pmatrix} a' \\ b' \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & q \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \tag{2.12}$$

and scaling, if it is necessary, we find the following inequivalent LS:

$$(2A_{1,1} + 3A)^1 : \ [\alpha, \alpha] = a, \quad [\beta, \beta] = b, \quad [\gamma, \gamma] = a + b;$$

$$(2A_{1,1} + 3A)^2 : \ [\alpha, \alpha] = a, \quad [\beta, \beta] = b, \quad [\gamma, \gamma] = a - b;$$

$$(2A_{1,1} + 3A)^3 : \ [\alpha, \alpha] = a, \quad [\beta, \beta] = b, \quad [\gamma, \gamma] = -(a + b).$$

In case (2), we use a linear transformation of the form:

$$\begin{pmatrix} \alpha' \\ \beta' \\ \gamma' \end{pmatrix} = \begin{pmatrix} Q & \mathrm{O}_1 \\ \mathrm{O}_2 & \mathrm{I}_1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}, \tag{2.13}$$

where Q is a 2×2 real orthogonal matrix diagonalizing T_1, i.e. $Q^T T_1 Q = \mathrm{diag}(p, q)$, $\mathrm{O}_1 = (0\ 0)^T$, $\mathrm{O}_2 = (0\ 0)$ and I_1 is the 1×1 unit matrix. Transformation

(2.13) does not affect the matrix S and finally, it leads to the pair:

$$S'' = \mathrm{diag}(1, 1, -1), \quad T'' = \begin{pmatrix} p & 0 & \kappa \\ 0 & q & \lambda \\ \kappa & \lambda & r \end{pmatrix}, \quad p, q, r, \kappa, \lambda \in \mathbb{R}.$$

In the sequel, we distinguish three basic cases:

$$\text{(i) } p \neq q, \quad \text{(ii) } p = q = -r, \quad \text{(iii) } p = q \neq -r,$$

which finally lead to the following inequivalent LS, where $p, q, \kappa, \lambda \in \mathbb{R}^*$:

$(2A_{1,1} + 3A)^9$: $[\alpha, \alpha] = a,\ [\beta, \beta] = b,\ [\gamma, \gamma] = pa - (p+1)b,$
$[\alpha, \gamma] = \kappa(a - b),\ [\beta, \gamma] = \lambda(a - b);$

$(2A_{1,1} + 3A)^{10}$: $[\alpha, \alpha] = a,\ [\beta, \beta] = a,\ [\gamma, \gamma] = -a,\ [\beta, \gamma] = b;$

$(2A_{1,1} + 3A)^{11}$: $[\alpha, \alpha] = a,\ [\beta, \beta] = a,\ [\gamma, \gamma] = -a,\ [\alpha, \gamma] = \kappa b, [\beta, \gamma] = b;$

$(2A_{1,1} + 3A)^{12}$: $[\alpha, \alpha] = a,\ [\beta, \beta] = a,\ [\gamma, \gamma] = b,\ [\alpha, \gamma] = \kappa(a + b);$

$(2A_{1,1} + 3A)^{13}$: $[\alpha, \alpha] = a,\ [\beta, \beta] = a,\ [\gamma, \gamma] = b,\ [\alpha, \gamma] = \kappa(a + b),$
$[\beta, \gamma] = \lambda(a + b);$

$(2A_{1,1} + 3A)^4$: $[\alpha, \alpha] = a,\ [\beta, \beta] = b,\ [\alpha, \gamma] = a - b.$

In the second case, transformations of the form (2.13), lead to the cases:

(3) $S' = \mathrm{diag}(1, 1, 0), \quad T' = (t'_{ij}),$
(4) $S' = \mathrm{diag}(1, -1, 0), \quad T' = (t'_{ij}),$
(5) $S' = \mathrm{diag}(1, 0, 0), \quad T' = (t'_{ij}),$

where (t_{ij}) is a 3×3 real symmetric matrix, with $\det T' = 0$.

In case (3), we use a linear transformation of the form:

$$\begin{pmatrix} \alpha' \\ \beta' \\ \gamma' \end{pmatrix} = P \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}, \quad \text{with } P = \begin{pmatrix} P_1 & O_1 \\ O_2 & P_2 \end{pmatrix},$$

where P_1 is an orthogonal 2×2 matrix diagonalizing $(t'_{ij})_{2\times 2}$ and P_2 is a 1×1 matrix with element $\frac{1}{\sqrt{|t'_{33}|}}$, if $t'_{33} \neq 0$ or 1, if $t'_{33} = 0$. Also, $O_1 = (0\ \ 0)^T$ and $O_2 = (0\ 0)$. So, we conclude the following pair of real symmetric matrices:

$$S' = \mathrm{diag}(1, 1, 0), \quad T' = \begin{pmatrix} p & 0 & \kappa \\ 0 & q & \lambda \\ \kappa & \lambda & \varepsilon \end{pmatrix},$$

where $\varepsilon \in \{0, 1, -1\}$ and $\det T' = 0$.

We distinguish cases as $p \neq q$ or $p = q$ and finally find the following inequivalent LS, where $\kappa, \lambda \in \mathbb{R}^*$:

$$(2A_{1,1}+3A)^5: \ [\alpha,\alpha]=a,\ [\beta,\beta]=b,\ [\alpha,\gamma]=a-b,\ [\beta,\gamma]=\lambda(a-b);$$

$$(2A_{1,1}+3A)^6: \ [\alpha,\alpha]=a,\ [\beta,\beta]=b,\ [\gamma,\gamma]=a-b,\ [\alpha,\gamma]=\kappa(a-b),\ [\beta,\gamma]=\lambda(a-b);$$

$$(2A_{1,1}+3A)^7: \ [\alpha,\alpha]=a,\ [\beta,\beta]=b,\ [\gamma,\gamma]=a-b,\ [\alpha,\beta]=\lambda(a-b).$$

In case (4), following the same method as in case (3), we find the LS:

$$(2A_{1,1}+3A)^{14}: \ [\alpha,\alpha]=a,\ [\beta,\beta]=b,\ [\alpha,\beta]=\kappa(a+b),\ [\alpha,\gamma]=a+b,\ [\beta,\gamma]=\mu(a+b);$$

$$(2A_{1,1}+3A)^{15}: \ [\alpha,\alpha]=a,\ [\beta,\beta]=b,\ [\alpha,\gamma]=a+b;$$

$$(2A_{1,1}+3A)^{16}: \ [\alpha,\alpha]=a,\ [\beta,\beta]=b,\ [\gamma,\gamma]=a+b,\ [\alpha,\beta]=\kappa(a+b),\ [\alpha,\gamma]=\lambda(a+b),\ [\beta,\gamma]=\mu(a+b);$$

$$(2A_{1,1}+3A)^{17}: \ [\alpha,\alpha]=a,\ [\beta,\beta]=b,\ [\gamma,\gamma]=-(a+b),\ [\alpha,\beta]=\kappa(a+b),\ [\alpha,\gamma]=\lambda(a+b),\ [\beta,\gamma]=\mu(a+b);$$

$$(2A_{1,1}+3A)^{18}: \ [\alpha,\alpha]=a,\ [\beta,\beta]=-a,\ [\gamma,\gamma]=b,\ [\alpha,\beta]=\kappa b,\ [\alpha,\gamma]=b,\ [\beta,\gamma]=\mu b;$$

$$(2A_{1,1}+3A)^{19}: \ [\alpha,\alpha]=a,\ [\beta,\beta]=-a,\ [\gamma,\gamma]=b,\ [\alpha,\beta]=b,\ [\beta,\gamma]=\mu b;$$

$$(2A_{1,1}+3A)^{20}: \ [\alpha,\alpha]=a,\ [\beta,\beta]=-a,\ [\gamma,\gamma]=b,\ [\beta,\gamma]=b;$$

$$(2A_{1,1}+3A)^{21}: \ [\alpha,\alpha]=a,\ [\beta,\beta]=-a,\ [\alpha,\beta]=b,\ [\alpha,\gamma]=b,\ [\beta,\gamma]=\mu b;$$

$$(2A_{1,1}+3A)^{22}: \ [\alpha,\alpha]=a,\ [\beta,\beta]=-a,\ [\beta,\gamma]=b;$$

$$(2A_{1,1}+3A)^{23}: \ [\alpha,\beta]=a,\ [\alpha,\gamma]=b.$$

Also, for the case (5), working as above, we obtain the following LS:

$$(2A_{1,1}+3A)^8: \ [\alpha,\alpha]=a,\ [\beta,\beta]=b,\ [\alpha,\gamma]=b.$$

Remark 2.2. The following tabulations are of families of equivalence classes of real indecomposable Lie superalgebras of dimension 5, which are not Lie algebras. For the labelling of the 5-dimensional LS, we use the letter E. The symbols a, b, c, d instead of a_1, a_2, a_3, a_4 and $\alpha, \beta, \gamma, \delta$ instead of $\alpha_1, \alpha_2, \alpha_3, \alpha_4$, make the use of the next tables easier.

Table I. (4,1)-Lie Superalgebras

	Relations	Comments
Trivial LS		
E^1_{pqr}	$[d,a]=pa, [d,b]=qb, [d,c]=rc; [d,\alpha]=\alpha$	$p,q,r\neq 0$
E^2_{pq}	$[a,b]=c; [d,a]=pa, [d,b]=qb, [d,c]=(p+q)c; [d,\alpha]=\alpha$	$p,q\neq 0$
$E^3_{\mu r}$	$[a,b]=b, [a,c]=\mu c; [d,c]=rc; [d,\alpha]=\alpha$	$r\neq 0,\ \mu\neq 0,1$
E^4_{q}	$[a,b]=b, [a,c]=b+c; [d,b]=qb, [d,c]=qc; [d,\alpha]=\alpha$	$q\neq 0$
E^5_{q}	$[a,b]=c; [d,b]=qb, [d,c]=qc; [d,\alpha]=\alpha$	$q\neq 0$
$E^6_{q\kappa}$	$[a,b]=\kappa b-c, [a,c]=b+\kappa c; [d,b]=qb, [d,c]=qc; [d,\alpha]=\alpha$	$q,\kappa\neq 0$
E^7_{q}	$[a,b]=-c, [a,c]=b; [d,b]=qb, [d,c]=qc; [d,\alpha]=\alpha$	$q\neq 0$
E^8_{p}	$[d,a]=pa, [d,b]=a+pb, [d,c]=b+pc; [d,\alpha]=\alpha$	$p\neq 0$
E^9_{λ}	$[a,c]=a, [b,c]=\lambda a+b; [d,b]=a, [d,c]=b; [d,\alpha]=\alpha$	$\lambda\neq 0$
E^{10}	$[a,c]=a, [b,c]=b; [d,b]=a, [d,c]=b; [d,\alpha]=\alpha$	
E^{11}_{κ}	$[b,c]=\kappa a; [d,b]=a, [d,c]=b; [d,\alpha]=\alpha$	$\kappa\neq 0$
E^{12}_{pq}	$[d,a]=pa, [d,b]=a+pb, [d,c]=qc; [d,\alpha]=\alpha$	$p,q\neq 0$
E^{13}_{q}	$[a,b]=c; [d,a]=pa, [d,b]=a+pb, [d,c]=2pc; [d,\alpha]=\alpha$	$p\neq 0$
E^{14}_{μ}	$[b,a]=a+c, [b,c]=\mu c; [d,b]=a; [d,\alpha]=\alpha$	$\mu\neq 0,\ \|\mu\|\leq 1$
$E^{15}_{\kappa\lambda}$	$[b,a]=\kappa a-\lambda c, [b,c]=\lambda a+\kappa c; [d,b]=a; [d,\alpha]=\alpha$	$\lambda\neq 0$
E^{16}_{pqr}	$[d,a]=pa, [d,b]=qb-rc, [d,c]=rb+qc; [d,\alpha]=\alpha$	$p,r\neq 0$
E^{17}_{qr}	$[b,c]=a; [d,a]=2qa, [d,b]=qb-rc, [d,c]=rb+qc;$ $[d,\alpha]=\alpha$	$r\neq 0$
E^{18}_{r}	$[a,b]=b, [a,c]=c; [d,b]=-rc, [d,c]=rb; [d,\alpha]=\alpha$	$r\neq 0$
E^{19}_{r}	$[b,c]=a; [d,b]=-rc, [d,c]=rb; [d,\alpha]=\alpha$	$r\neq 0$
Non trivial LS		
E^1_{2qr}	$[d,a]=2a, [d,b]=qb, [d,c]=rc; [d,\alpha]=\alpha; [\alpha,\alpha]=a$	$q,r\neq 2$
$E^2_{p,2-p}$	$[a,b]=c; [d,a]=pa, [d,b]=(2-p)b, [d,c]=2c;$ $[d,\alpha]=\alpha; [\alpha,\alpha]=c$	$p\neq 0,\ q\neq 2$
E^5_{2}	$[a,b]=c; [d,b]=2b, [d,c]=2c; [d,\alpha]=\alpha; [\alpha,\alpha]=c$	
E^8_{2}	$[d,a]=2a, [d,b]=a+2b, [d,c]=b+2c; [d,\alpha]=\alpha;$ $[\alpha,\alpha]=a$	
E^{12}_{2q}	$[d,a]=2a, [d,b]=a+2b, [d,c]=qc; [d,\alpha]=\alpha; [\alpha,\alpha]=c$	$q\neq 0,2$
E^{12}_{p2}	$[d,a]=pa, [d,b]=a+pb, [d,c]=2c; [d,\alpha]=\alpha; [\alpha,\alpha]=c$	$p\neq 0,2$
$\left(E^{13}_{1}\right)^1$	$[a,b]=c; [d,a]=a, [d,b]=a+b, [d,c]=2c; [d,\alpha]=\alpha;$ $[\alpha,\alpha]=c$	
$\left(E^{13}_{1}\right)^2$	$[a,b]=b; [d,a]=a, [d,b]=a+b, [d,c]=2c; [d,\alpha]=\alpha;$ $[\alpha,\alpha]=-c$	
E^{16}_{2r}	$[d,a]=2a, [d,b]=qb-rc, [d,c]=rb+qc; [d,\alpha]=\alpha;$ $[\alpha,\alpha]=a$	$r\neq 0$
$\left(E^{17}_{1r}\right)^1$	$[b,c]=a; [d,a]=2a, [d,b]=b-rc, [d,c]=rb+c; [d,\alpha]=\alpha;$ $[\alpha,\alpha]=a$	$r\neq 0$
$\left(E^{17}_{1r}\right)^2$	$[b,c]=a; [d,a]=2a, [d,b]=b-rc, [d,c]=rb+c;$ $[d,\alpha]=\alpha; [\alpha,\alpha]=-a$	$r\neq 0$
$(A_{4,3}+A)$	$[d,a]=2a, [d,c]=b; [\alpha,\alpha]=b$	
$(A_{4,1}+A)$	$[d,b]=a, [d,c]=b; [\alpha,\alpha]=a$	
$(A_{4,8}+A)^1$	$[a,b]=c; [d,a]=a, [d,b]=-b; [\alpha,\alpha]=c$	
$(A_{4,8}+A)^2$	$[a,b]=c; [d,a]=-b, [d,b]=a; [\alpha,\alpha]=c$	
$(A_{4,8}+A)^3$	$[a,b]=c; [d,a]=-b, [d,b]=a; [\alpha,\alpha]=-c$	

Table II. (1,4)-Lie Superalgebras

	Relations	Comments
Trivial LS		
E^{20}_{qrs}	$[a,\alpha]=\alpha, [a,\beta]=q\beta, [a,\gamma]=r\gamma, [a,\delta]=s\delta$	$0\le\lvert s\rvert\le\lvert r\rvert\le\lvert q\rvert\le 1$
E^{21}_{p}	$[a,\alpha]=p\alpha+\beta, [a,\beta]=p\beta+\gamma, [a,\gamma]=p\gamma+\delta,$	
	$[a,\delta]=p\delta$	$p\neq 0$
E^{22}	$[a,\alpha]=\beta, [a,\beta]=\gamma, [a,\gamma]=\delta$	
E^{23}_{p}	$[a,\alpha]=p\alpha, [a,\beta]=p\beta+\gamma, [a,\gamma]=p\gamma+\delta, [a,\delta]=p\delta$	$p\neq 0$
E^{24}_{p}	$[a,\alpha]=p\alpha+\beta, [a,\beta]=p\beta, [a,\gamma]=p\gamma+\delta, [a,\delta]=p\delta$	$p\neq 0$
E^{25}	$[a,\alpha]=\beta, [a,\gamma]=\delta$	
E^{26}_{p}	$[a,\alpha]=p\alpha, [a,\beta]=p\beta, [a,\gamma]=p\gamma+\delta, [a,\delta]=p\delta$	$p\neq 0$
E^{27}_{pq}	$[a,\alpha]=p\alpha, [a,\beta]=q\beta+\gamma, [a,\gamma]=q\gamma+\delta, [a,\delta]=q\delta$	$p,q\neq 0,\ p\neq q$
E^{28}_{p}	$[a,\alpha]=p\alpha, [a,\beta]=\gamma, [a,\gamma]=\delta$	$p\neq 0$
E^{29}_{pq}	$[a,\alpha]=p\alpha, [a,\beta]=q\beta, [a,\gamma]=q\gamma+\delta, [a,\delta]=q\delta$	$p,q\neq 0,\ p\neq q$
E^{30}_{pq}	$[a,\alpha]=p\alpha+\beta, [a,\beta]=p\beta, [a,\gamma]=q\gamma+\delta, [a,\delta]=q\delta$	$p\neq q$
E^{31}_{pq}	$[a,\alpha]=p\alpha, [a,\beta]=p\beta, [a,\gamma]=q\gamma+\delta, [a,\delta]=q\delta$	$p\neq 0,\ p\neq q$
E^{32}_{pqs}	$[a,\alpha]=p\alpha+q\beta, [a,\beta]=-q\alpha+p\beta, [a,\gamma]=r\gamma+s\delta,$	
	$[a,\delta]=-s\gamma+r\delta$	$q,s\neq 0$
E^{33}_{pqs}	$[a,\alpha]=p\alpha, [a,\beta]=q\beta, [a,\gamma]=r\gamma+s\delta,$	
	$[a,\delta]=-s\gamma+r\delta$	$s\neq 0,\ \lvert p\rvert\ge\lvert q\rvert>0$
E^{34}_{s}	$[a,\alpha]=p\alpha+\beta, [a,\beta]=p\beta, [a,\gamma]=r\gamma+s\delta,$	
	$[a,\delta]=-s\gamma+r\delta$	$s\neq 0$
Non trivial LS		
$(A_{1,1}+4A)^1$	$[\alpha,\alpha]=a, [\beta,\beta]=a, [\gamma,\gamma]=a, [\delta,\delta]=a$	
$(A_{1,1}+4A)^2$	$[\alpha,\alpha]=a, [\beta,\beta]=a, [\gamma,\gamma]=a, [\delta,\delta]=-a$	
$(A_{1,1}+4A)^3$	$[\alpha,\alpha]=a, [\beta,\beta]=a, [\gamma,\gamma]=-a, [\delta,\delta]=-a$	

Table III. (3,2)-Lie Superalgebras

	Relations	Comments
Trivial LS		
$E^{35}_{\lambda\mu\nu}$	$[a,b]=2b,[a,c]=-2c,[b,c]=a;[a,\alpha]=(\lambda+2)\alpha,$	
	$[a,\beta]=\lambda\beta,[b,\beta]=\mu\alpha,[c,\alpha]=\nu\beta$	$\mu,\nu\neq 0$
E^{36}	$[a,b]=2b,[a,c]=-2c,[b,c]=a;[a,\alpha]=\alpha,[a,\beta]=-\beta,$	
	$[a,\beta]=-\beta,[b,\beta]=\alpha,[c,\alpha]=\beta$	*full BRS algebra*
E^{37}	$[a,b]=b;[a,\alpha]=\alpha,[b,\beta]=\alpha,[c,\alpha]=\alpha,[c,\beta]=\beta$	
E^{38}_{λ}	$[a,b]=b;[a,\beta]=\lambda\beta,[c,\alpha]=\alpha,[c,\beta]=\lambda\beta$	$\lambda\neq 0$
$E^{39}_{\lambda\mu}$	$[a,b]=b;[a,\alpha]=\lambda\alpha,[a,\beta]=\lambda\beta,[c,\alpha]=\alpha+\mu\beta,[c,\beta]=\beta$	$\lambda,\mu\neq 0$
E^{40}_{λ}	$[a,b]=b;[a,\alpha]=\lambda\alpha,[a,\beta]=\lambda\beta,[c,\alpha]=\beta$	$\lambda\neq 0$
E^{41}_{λ}	$[a,b]=b;[a,\alpha]=(1-\lambda)\beta,[c,\alpha]=\alpha+\lambda\beta,[c,\beta]=\beta$	
$E^{42}_{p\lambda}$	$[a,b]=b,[a,c]=pc;[a,\alpha]=(\lambda+1)\alpha,[a,\beta]=\lambda\beta,[b,\beta]=\alpha$	$0\leq\|p\|\leq 1,\ \lambda\in\mathbb{R}$
$E^{43}_{p\lambda\mu}$	$[a,b]=b,[a,c]=pc;[a,\alpha]=\lambda\alpha,[a,\beta]=\mu\beta$	$\|p\|\leq 1,\ \ p,\lambda,\mu\neq 0$
$E^{44}_{p\lambda}$	$[a,b]=b,[a,c]=pc;[a,\alpha]=\lambda\alpha,[a,\beta]=\alpha+\lambda\beta$	$0<\|p\|\leq 1,\ \lambda\neq 0$
E^{45}_{p}	$[a,b]=b,[a,c]=pc;[a,\beta]=\alpha$	$0<\|p\|\leq 1$
E^{46}_{λ}	$[a,b]=b,[a,c]=b+c;[a,\alpha]=(\lambda+1)\alpha,[a,\beta]=\lambda\beta,[c,\beta]=\alpha$	$\lambda\in\mathbb{R}$
$E^{47}_{\lambda,\mu}$	$[a,b]=b,[a,c]=b+c;[a,\alpha]=\lambda\alpha,[a,\beta]=\mu\beta$	$\lambda,\mu\neq 0$
E^{48}_{λ}	$[a,b]=b,[a,c]=b+c;[a,\alpha]=\lambda\alpha,[a,\beta]=\alpha+\lambda\beta$	$\lambda\neq 0$
E^{49}	$[a,b]=b,[a,c]=b+c;[a,\beta]=\alpha$	
E^{50}_{pq}	$[a,c]=pc,[b,c]=qc;[a,\alpha]=\alpha,[a,\beta]=\beta,[b,\beta]=\alpha$	$p,q\neq 0$
E^{51}_{pq}	$[a,c]=pc,[b,c]=qc;[a,\alpha]=\alpha,[a,\beta]=\beta,$	
	$[b,\alpha]=-\alpha,[b,\beta]=\beta$	$\|p\|+\|q\|\neq 0$
E^{52}_{pq}	$[a,c]=pc,[b,c]=qc;[a,\alpha]=\alpha,[a,\beta]=\beta,$	
	$[b,\alpha]=-\beta,[b,\beta]=\alpha$	$\|p\|+\|q\|\neq 0$
E^{53}	$[a,b]=c;[a,\alpha]=\alpha,[a,\beta]=\beta,[b,\beta]=\alpha$	
E^{54}	$[a,b]=c;[a,\alpha]=\alpha,[b,\beta]=\beta$	
$E^{55}_{\lambda\mu}$	$[a,b]=c;[a,\alpha]=\lambda\alpha,[a,\beta]=\mu\beta$	$\lambda,\mu\neq 0$

Table III. Continued

	Relations	Comments
E^{56}_{λ}	$[a,b]=c; [a,\alpha]=\lambda\alpha, [a,\beta]=\alpha+\lambda\beta$	$\lambda\neq 0$
E^{57}	$[a,b]=c; [a,\beta]=\alpha$	
E^{58}_{λ}	$[a,b]=c; [a,\alpha]=\lambda\alpha+\beta, [a,\beta]=\lambda\beta,$	
	$[b,\alpha]=\beta, [c,\alpha]=\alpha, [c,\beta]=\beta$	$\lambda\neq 0$
E^{59}	$[a,b]=c; [a,\alpha]=\beta, [c,\alpha]=\alpha, [c,\beta]=\beta$	
E^{60}	$[a,b]=c; [a,\alpha]=\alpha, [a,\beta]=\beta, [b,\alpha]=-\beta, [b,\beta]=\alpha$	
E^{61}_{pqr}	$[a,b]=b, [a,c]=pc; [a,\alpha]=q\alpha-r\beta, [a,\beta]=r\alpha+q\beta$	$0<\lvert p\rvert\leq 1, r\neq 0$
E^{62}_{qr}	$[a,b]=b, [a,c]=b+c; [a,\alpha]=q\alpha-r\beta, [a,\beta]=r\alpha+q\beta$	$r\neq 0$
E^{63}_{qr}	$[a,b]=c; [a,\alpha]=q\alpha-r\beta, [a,\beta]=r\alpha+q\beta$	$r\neq 0$
E^{64}_{pqr}	$[a,b]=pb-c, [a,c]=b+pc; [a,\alpha]=q\alpha, [a,\beta]=r\beta$	$q,r\neq 0$
E^{65}_{pq}	$[a,b]=pb-c, [a,c]=b+pc; [a,\alpha]=q\alpha, [a,\beta]=\alpha+r\beta$	$q\neq 0$
E^{66}_{p}	$[a,b]=pb-c, [a,c]=b+pc; [a,\beta]=\alpha$	
E^{67}_{pqr}	$[a,b]=pb-c, [a,c]=b+pc; [a,\alpha]=q\alpha-r\beta,$	
	$[a,\beta]=r\alpha+q\beta$	
Non trivial LS		
E^{36}	$[a,b]=b, [a,c]=-c, [b,c]=2a; [a,\alpha]=\frac{1}{2}\alpha, [a,\beta]=-\frac{1}{2}\beta,$	*di-spin*
	$[b,\beta]=\frac{1}{2}\alpha, [c,\alpha]=\frac{1}{2}\beta; [\alpha,\alpha]=b, [\beta,\beta]=-c, [\alpha,\beta]=-a$	*or* $osp(1,2)$
$(E^{37})^1$	$[a,b]=b; [a,\alpha]=\alpha, [b,\beta]=\alpha; [\beta,\beta]=c$	
$(E^{37})^2$	$[a,b]=b; [a,\alpha]=\frac{3}{2}\alpha, [a,\beta]=\frac{1}{2}\beta, [b,\beta]=\alpha; [\beta,\beta]=c$	
$E^{42}_{p\frac{p}{2}}$	$[a,b]=b, [a,c]=pc; [a,\alpha]=(\frac{p}{2}+1)\alpha, [a,\beta]=\frac{p}{2}\beta,$	
	$[b,\beta]=\alpha; [\beta,\beta]=c$	$0\leq\lvert p\rvert\leq 1$
$\left(E^{43}_{p\frac{1}{2}\frac{1}{2}}\right)^1$	$[a,b]=b, [a,c]=pc; [a,\alpha]=\frac{1}{2}\alpha, [a,\beta]=\frac{1}{2}\beta; [\alpha,\alpha]=b$	$p\neq 0$
$\left(E^{43}_{p\frac{1}{2}\frac{1}{2}}\right)^2$	$[a,b]=b, [a,c]=pc; [a,\alpha]=\frac{1}{2}\alpha, [a,\beta]=\frac{1}{2}\beta;$	
	$[\alpha,\alpha]=b, [\beta,\beta]=b$	$p\neq 0$
$\left(E^{43}_{p\frac{1}{2}\frac{1}{2}}\right)^3$	$[a,b]=b, [a,c]=pc; [a,\alpha]=\frac{1}{2}\alpha, [a,\beta]=\frac{1}{2}\beta;$	
	$[\alpha,\alpha]=b, [\beta,\beta]=-b$	$p\neq 0$

Table III. Continued

	Relations	Comments
$E^{43}_{p\frac{1}{2}\frac{p}{2}}$	$[a, b] = b, [a, c] = pc; [a, \alpha] = \frac{1}{2}\alpha, [a, \beta] = \frac{p}{2}\beta;$ $[\alpha, \alpha] = b, [\beta, \beta] = c$	$p \neq 0$
$E^{43}_{p\frac{1}{2},p-\frac{1}{2}}$	$[a, b] = b, [a, c] = pc; [a, \alpha] = \frac{1}{2}\alpha, [a, \beta] = (p - \frac{1}{2})\beta;$ $[\alpha, \alpha] = b, [\alpha, \beta] = c$	
$E^{43}_{1\frac{1}{2}r}$	$[a, b] = b, [a, c] = c; [a, \alpha] = \frac{1}{2}\alpha, [a, \beta] = r\beta; [\alpha, \alpha] = b$	$r \neq 0, \frac{1}{2}$
$E^{43}_{1q,1-q}$	$[a, b] = b, [a, c] = c; [a, \alpha] = q\alpha, [a, \beta] = (1 - q)\beta; [\alpha, \beta] = b$	$q \neq \frac{1}{2}$
$E^{44}_{p\frac{1}{2}}$	$[a, b] = b, [a, c] = pc; [a, \alpha] = \frac{1}{2}\alpha, [a, \beta] = \alpha + \frac{1}{2}\beta; [\beta, \beta] = b$	$p \neq 0$
$E^{46}_{\frac{1}{2}}$	$[a, b] = b, [a, c] = b + c; [a, \alpha] = \frac{3}{2}\alpha, [a, \beta] = \frac{1}{2}\beta, [c, \beta] = \alpha;$ $[\beta, \beta] = b$	
$\left(E^{47}_{\frac{1}{2}\frac{1}{2}}\right)^1$	$[a, b] = b, [a, c] = b + c; [a, \alpha] = \frac{1}{2}\alpha, [a, \beta] = \frac{1}{2}\beta;$ $[\alpha, \alpha] = b, [\beta, \beta] = b$	
$\left(E^{47}_{\frac{1}{2}\frac{1}{2}}\right)^2$	$[a, b] = b, [a, c] = b + c; [a, \alpha] = \frac{1}{2}\alpha, [a, \beta] = \frac{1}{2}\beta;$ $[\alpha, \alpha] = b, [\beta, \beta] = -b$	
$E^{47}_{\frac{1}{2}r}$	$[a, b] = b, [a, c] = b + c; [a, \alpha] = \frac{1}{2}\alpha, [a, \beta] = r\beta; [\alpha, \alpha] = b$	$r \neq 0$
$\left(E^{48}_{\frac{1}{2}}\right)^1$	$[a, b] = b, [a, c] = b + c; [a, \alpha] = \frac{1}{2}\alpha, [a, \beta] = \alpha + \frac{1}{2}\beta;$ $[\beta, \beta] = b$	
$\left(E^{48}_{\frac{1}{2}}\right)^2$	$[a, b] = b, [a, c] = b + c; [a, \alpha] = \frac{1}{2}\alpha, [a, \beta] = \alpha + \frac{1}{2}\beta;$ $[\alpha, \beta] = \frac{1}{2}b, [\beta, \beta] = c$	
E^{50}_{20}	$[a, c] = 2c; [a, \alpha] = \alpha, [a, \beta] = \beta, [b, \beta] = \alpha; [\beta, \beta] = c$	
E^{51}_{22}	$[a, c] = 2c, [b, c] = 2c; [a, \alpha] = \alpha, [a, \beta] = \beta,$ $[b, \alpha] = -\alpha, [b, \beta] = \beta; [\beta, \beta] = c$	
E^{51}_{20}	$[a, c] = 2c; [a, \alpha] = \alpha, [a, \beta] = \beta, [b, \alpha] = -\alpha, [b, \beta] = \beta;$ $[\alpha, \beta] = c$	
$E^{55}_{1,-1}$	$[a, b] = c; [a, \alpha] = \alpha, [a, \beta] = -\beta; [\alpha, \beta] = c$	

Table III. Continued

	Relations	Comments
E^{55}_{10}	$[a,b]=c, [a,\alpha]=\alpha, [\beta,\beta]=c$	
$\left(E^{55}_{00}\right)^1$	$[a,b]=c, [\alpha,\alpha]=c, [\beta,\beta]=c$	*Heisenberg algebra*
$\left(E^{55}_{00}\right)^2$	$[a,b]=c, [\alpha,\alpha]=c, [\beta,\beta]=-c$	
E^{57}_{00}	$[a,b]=c, [a,\beta]=\alpha, [\beta,\beta]=c$	
$E^{61}_{p\frac{1}{2}r}$	$[a,b]=b, [a,c]=pc, [a,\alpha]=\frac{1}{2}\alpha-r\beta, [a,\beta]=r\alpha+\frac{1}{2}\beta,$	
	$[\alpha,\alpha]=b, [\beta,\beta]=b$	
$E^{62}_{\frac{1}{2}r}$	$[a,b]=b, [a,c]=b+c, [a,\alpha]=\frac{1}{2}\alpha-r\beta, [a,\beta]=r\alpha+\frac{1}{2}\beta,$	
	$[\alpha,\alpha]=b, [\beta,\beta]=b,$	$r\neq 0$
E^{63}_{01}	$[a,b]=c, [a,\alpha]=-\beta, [a,\beta]=\alpha, [\alpha,\alpha]=c, [\beta,\beta]=c$	
$E^{67}_{p\frac{p}{2}\frac{1}{2}}$	$[a,b]=pb-c, [a,c]=b+pc, [a,\alpha]=\frac{1}{2}(p\alpha-\beta),$	
	$[a,\beta]=\frac{1}{2}(\alpha+p\beta), [\alpha,\alpha]=b, [\beta,\beta]=-b, [\alpha,\beta]=c$	
$3A_{1,1}+2A$	$[\alpha,\alpha]=a, [\beta,\beta]=b, [\alpha,\beta]=c$	

Table IV. (2,3)-Lie Superalgebras

	Relations	Comments
Trivial LS		
E^{68}_{pqr}	$[a,b]=b; [a,\alpha]=p\alpha, [a,\beta]=q\beta, [a,\gamma]=r\gamma$	$\vert p\vert \geq \vert q\vert \geq \vert r\vert > 0$
E^{69}_{pq}	$[a,b]=b; [a,\alpha]=p\alpha, [a,\beta]=(q+1)\beta, [a,\gamma]=q\gamma, [b,\gamma]=\beta$	$p \neq 0$
E^{70}_{p}	$[a,b]=b; [a,\alpha]=(p+2)\alpha, [a,\beta]=(p+1)\beta, [a,\gamma]=p\gamma,$	
	$[b,\beta]=\alpha, [b,\gamma]=\beta$	$p \in \mathbb{R}$
E^{71}_{pq}	$[a,b]=b; [a,\alpha]=p\alpha, [a,\beta]=\alpha+p\beta, [a,\gamma]=q\gamma$	$q \neq 0,\ p \in \mathbb{R}$
E^{72}_{p}	$[a,b]=b; [a,\alpha]=p\alpha, [a,\beta]=\alpha+p\beta, [a,\gamma]=(p-1)\gamma, [b,\gamma]=\alpha$	$p \in \mathbb{R}$
E^{73}_{p}	$[a,b]=b; [a,\alpha]=p\alpha, [a,\beta]=\alpha+p\beta, [a,\gamma]=(p+1)\gamma, [b,\beta]=\gamma$	$p \in \mathbb{R}$
E^{74}_{p}	$[a,b]=b; [a,\alpha]=p\alpha, [a,\beta]=\alpha+p\beta, [a,\gamma]=b+p\gamma$	$p \in \mathbb{R}$
E^{75}_{pqr}	$[a,\alpha]=\alpha, [a,\beta]=p\beta, [a,\gamma]=q\gamma, [b,\beta]=\beta, [b,\gamma]=r\gamma$	$1 \geq \vert p\vert \geq \vert r\vert > 0$
E^{76}_{p}	$[a,\alpha]=\alpha, [a,\beta]=\beta, [a,\gamma]=p\gamma, [b,\beta]=\alpha, [b,\gamma]=\gamma$	$p \in \mathbb{R}$
E^{77}	$[a,\alpha]=\alpha, [a,\beta]=\beta, [a,\gamma]=\gamma, [b,\beta]=\alpha, [b,\gamma]=\beta$	
E^{78}_{pq}	$[a,\alpha]=\alpha, [a,\beta]=p\alpha+\beta, [a,\gamma]=q\gamma, [b,\beta]=\alpha, [b,\gamma]=\gamma$	$p \neq 0,\ q \in \mathbb{R}$
E^{79}_{pq}	$[a,\alpha]=\alpha, [a,\beta]=p\alpha+\beta, [a,\gamma]=q\gamma, [b,\alpha]=\gamma$	$p \neq 0,\ q \in \mathbb{R}$
E^{80}	$[a,\beta]=\alpha, [b,\alpha]=\gamma$	
E^{81}_{pq}	$[a,\alpha]=\alpha, [a,\beta]=p\alpha+\beta, [a,\gamma]=p\beta+\gamma, [b,\beta]=\alpha,$	
	$[b,\gamma]=q\alpha+\beta$	$p, q \neq 0$
E^{82}_{p}	$[a,\alpha]=\alpha, [a,\beta]=p\alpha+\beta, [a,\gamma]=p\beta+\gamma, [b,\gamma]=\alpha$	$p \neq 0$
E^{83}	$[a,\beta]=\alpha, [a,\gamma]=\beta, [b,\gamma]=\beta$	
E^{84}_{pqr}	$[a,b]=b; [a,\alpha]=p\alpha-r\beta, [a,\beta]=r\alpha+p\beta, [a,\gamma]=q\gamma$	$q, r \neq 0$
E^{85}_{p}	$[a,\alpha]=\alpha, [a,\beta]=\gamma, [a,\gamma]=-\beta, [b,\alpha]=p\alpha, [b,\beta]=\beta, [b,\gamma]=\gamma$	
Non trivial LS		
$E^{68}_{\frac{1}{2}qr}$	$[a,b]=b; [a,\alpha]=\frac{1}{2}\alpha, [a,\beta]=q\beta, [a,\gamma]=r\gamma; [\alpha,\alpha]=b$	$q, r \neq 0$
$E^{68}_{\frac{1}{2}\frac{1}{2}r}$	$[a,b]=b; [a,\alpha]=\frac{1}{2}\alpha, [a,\beta]=\frac{1}{2}\beta, [a,\gamma]=r\gamma;$	

Table IV. Continued

	Relations	Comments
	$[\alpha,\alpha]=b,[\beta,\beta]=b$	$r\neq 0$
$E^{68}_{\frac{1}{2}\frac{1}{2}\frac{1}{2}}$	$[a,b]=b;[a,\alpha]=\frac{1}{2}\alpha,[a,\beta]=\frac{1}{2}\beta,[a,\gamma]=\frac{1}{2}\gamma;$	
	$[\alpha,\alpha]=b,[\beta,\beta]=b,[\gamma,\gamma]=b$	
$E^{68}_{\frac{1}{2}q,1-q}$	$[a,b]=b;[a,\alpha]=\frac{1}{2}\alpha,[a,\beta]=q\beta,[a,\gamma]=(1-q)\gamma;$	
	$[\alpha,\alpha]=b,[\beta,\gamma]=b$	$q\neq 0$
$\left(E^{69}_{00}\right)^1$	$[a,b]=b;[a,\beta]=\beta,[b,\gamma]=\beta;$	
	$[\gamma,\gamma]=-2a,[\beta,\gamma]=b,[\alpha,\gamma]=a,[\alpha,\beta]=-b$	
$\left(E^{69}_{00}\right)^2$	$[a,b]=b;[a,\beta]=\beta,[b,\gamma]=\beta;[\alpha,\gamma]=a,[\alpha,\beta]=-b$	
$E^{71}_{p,1-p}$	$[a,b]=b;[a,\alpha]=p\alpha,[a,\beta]=\alpha+p\beta,[a,\gamma]=(1-p)\gamma,$	
	$[\beta,\gamma]=b$	$p\in\mathbb{R}$
$E^{71}_{p\frac{1}{2}}$	$[a,b]=b;[a,\alpha]=p\alpha,[a,\beta]=\alpha+p\beta,[a,\gamma]=\frac{1}{2}\gamma,$	
	$[\gamma,\gamma]=b$	$p\in\mathbb{R}$
$\left(E^{71}_{\frac{1}{2}\frac{1}{2}}\right)^1$	$[a,b]=b;[a,\alpha]=\frac{1}{2}\alpha,[a,\beta]=\alpha+\frac{1}{2}\beta,[a,\gamma]=\frac{1}{2}\gamma,$	
	$[\beta,\beta]=b$	
$\left(E^{71}_{\frac{1}{2}\frac{1}{2}}\right)^2$	$[a,b]=b;[a,\alpha]=\frac{1}{2}\alpha,[a,\beta]=\alpha+\frac{1}{2}\beta,[a,\gamma]=\frac{1}{2}\gamma;$	
	$[\beta,\beta]=b,[\gamma,\gamma]=b$	
$\left(E^{74}_{\frac{1}{2}}\right)^1$	$[a,b]=b;[a,\alpha]=\frac{1}{2}\alpha,[a,\beta]=\alpha+\frac{1}{2}\beta,[a,\gamma]=\beta+\frac{1}{2}\gamma;$	
	$[\beta,\beta]=b,[\alpha,\gamma]=-b,[\gamma,\gamma]=\kappa b$	$\kappa\neq 0$
$\left(E^{74}_{\frac{1}{2}}\right)^2$	$[a,b]=b;[a,\alpha]=\frac{1}{2}\alpha,[a,\beta]=\alpha+\frac{1}{2}\beta,[a,\gamma]=\beta+\frac{1}{2}\gamma;$	
	$[\beta,\beta]=b,[\alpha,\gamma]=-b$	
$\left(E^{74}_{\frac{1}{2}}\right)^3$	$[a,b]=b;[a,\alpha]=\frac{1}{2}\alpha,[a,\beta]=\alpha+\frac{1}{2}\beta,[a,\gamma]=\beta+\frac{1}{2}\gamma;$	
	$[\gamma,\gamma]=b$	
$E^{84}_{p\frac{1}{2}r}$	$[a,b]=b;[a,\alpha]=p\alpha-r\beta,[a,\beta]=r\alpha+p\beta,[a,\gamma]=\frac{1}{2}\gamma;$	
	$[\gamma,\gamma]=b$	

Table IV. (2,3)-Lie Superalgebras

	Relations	Comments
$E^{84}_{\frac{1}{2}qr}$	$[a,b]=b;[a,\alpha]=\frac{1}{2}\alpha-r\beta,[a,\beta]=r\alpha+\frac{1}{2}\beta,[a,\gamma]=q\gamma;$ $[\alpha,\alpha]=b,[\beta,\beta]=b$	
$\left(E^{84}_{\frac{1}{2}\frac{1}{2}r}\right)^1$	$[a,b]=b;[a,\alpha]=\frac{1}{2}\alpha-r\beta,[a,\beta]=r\alpha+\frac{1}{2}\beta,[a,\gamma]=\frac{1}{2}\gamma;$ $[\alpha,\alpha]=b,[\beta,\beta]=b,[\gamma,\gamma]=b$	$r\neq 0$
$\left(E^{84}_{\frac{1}{2}\frac{1}{2}r}\right)^2$	$[a,b]=b;[a,\alpha]=\frac{1}{2}\alpha-r\beta,[a,\beta]=r\alpha+\frac{1}{2}\beta,[a,\gamma]=\frac{1}{2}\gamma;$ $[\alpha,\alpha]=b,[\beta,\beta]=b,[\gamma,\gamma]=-b$	$r\neq 0$
$D^{11}_{-1,0}+A$	$[a,\alpha]=\alpha,[a,\beta]=-\beta;[\alpha,\beta]=b,[\gamma,\gamma]=b$	
$D^{11}_{-1,q}+A$	$[a,\alpha]=\alpha,[a,\beta]=-\beta,[a,\gamma]=q\gamma;[\alpha,\beta]=b$	$q\neq 0,\pm 1$
$(C^3+A)^1$	$[a,\beta]=\alpha;[\beta,\beta]=b,[\gamma,\gamma]=b$	
$(C^3+A)^2$	$[a,\beta]=\alpha;[\beta,\beta]=b,[\gamma,\gamma]=-b$	
$(D^{15}+A_{1,1})^1$	$[a,\beta]=\alpha,[a,\gamma]=\beta;[\gamma,\gamma]=b$	
$(D^{15}+A_{1,1})^2$	$[a,\beta]=\alpha,[a,\gamma]=\beta;[\beta,\beta]=b,[\alpha,\gamma]=-b$	
$(D^{15}+A_{1,1})^3$	$[a,\beta]=\alpha,[a,\gamma]=\beta;[\beta,\beta]=b,[\gamma,\gamma]=b,[\alpha,\gamma]=-b$	
$(D^{15}+A_{1,1})^4$	$[a,\beta]=\alpha,[a,\gamma]=\beta;[\beta,\beta]=b,[\gamma,\gamma]=-b,[\alpha,\gamma]=-b$	
$(D^{14}_{00}+A_{1,1})^1$	$[a,\alpha]=-\beta,[a,\beta]=\alpha,[a,\gamma]=q\gamma;[\alpha,\alpha]=b,[\beta,\beta]=b$	
$(D^{14}_{00}+A_{1,1})^2$	$[a,\alpha]=-\beta,[a,\beta]=\alpha;[\alpha,\alpha]=b,[\beta,\beta]=b,[\gamma,\gamma]=b$	
$(D^{14}_{00}+A_{1,1})^3$	$[a,\alpha]=-\beta,[a,\beta]=\alpha;[\alpha,\alpha]=b,[\beta,\beta]=b,[\gamma,\gamma]=-b$	
$(2A_{1,1}+3A)^1$	$[\alpha,\alpha]=a,[\beta,\beta]=b,[\gamma,\gamma]=a+b$	
$(2A_{1,1}+3A)^2$	$[\alpha,\alpha]=a,[\beta,\beta]=b,[\gamma,\gamma]=a-b$	
$(2A_{1,1}+3A)^3$	$[\alpha,\alpha]=a,[\beta,\beta]=b,[\gamma,\gamma]=-(a+b)$	
$(2A_{1,1}+3A)^4$	$[\alpha,\alpha]=a,[\beta,\beta]=b,[\alpha,\gamma]=a-b$	
$(2A_{1,1}+3A)^5$	$[\alpha,\alpha]=a,[\beta,\beta]=b,[\alpha,\gamma]=a-b,[\beta,\gamma]=\lambda(a-b)$	$\lambda\neq 0$

Table IV. Continued

$(2A_{1,1}+3A)^6$	$[\alpha,\alpha]=a, [\beta,\beta]=b, [\gamma,\gamma]=a-b, [\alpha,\gamma]=\kappa(a-b),$ $[\beta,\gamma]=\lambda(a-b)$	$\kappa,\lambda\neq 0$
$(2A_{1,1}+3A)^7$	$[\alpha,\alpha]=a, [\beta,\beta]=b, [\gamma,\gamma]=a-b, [\alpha,\beta]=\lambda(a-b)$	$\lambda\neq 0$
$(2A_{1,1}+3A)^8$	$[\alpha,\alpha]=a, [\beta,\beta]=b, [\alpha,\gamma]=b$	
$(2A_{1,1}+3A)^9$	$[\alpha,\alpha]=a, [\beta,\beta]=b, [\gamma,\gamma]=pa-(p+1)b,$ $[\alpha,\gamma]=\kappa(a-b), [\beta,\gamma]=\lambda(a-b)$	
$(2A_{1,1}+3A)^{10}$	$[\alpha,\alpha]=a, [\beta,\beta]=a, [\gamma,\gamma]=-a, [\beta,\gamma]=b$	
$(2A_{1,1}+3A)^{11}$	$[\alpha,\alpha]=a, [\beta,\beta]=a, [\gamma,\gamma]=-a, [\alpha,\gamma]=\kappa b, [\beta,\gamma]=b$	$\kappa\neq 0$
$(2A_{1,1}+3A)^{12}$	$[\alpha,\alpha]=a, [\beta,\beta]=a, [\gamma,\gamma]=b, [\alpha,\gamma]=\kappa(a+b)$	$\kappa\neq 0$
$(2A_{1,1}+3A)^{13}$	$[\alpha,\alpha]=a, [\beta,\beta]=a, [\gamma,\gamma]=b, [\alpha,\gamma]=\kappa(a+b),$ $[\beta,\gamma]=\lambda(a+b)$	$\kappa,\lambda\neq 0$
$(2A_{1,1}+3A)^{14}$	$[\alpha,\alpha]=a, [\beta,\beta]=b, [\alpha,\beta]=\kappa(a+b), [\alpha,\gamma]=a+b,$ $[\beta,\gamma]=\mu(a+b)$	
$(2A_{1,1}+3A)^{15}$	$[\alpha,\alpha]=a, [\beta,\beta]=b, [\alpha,\gamma]=a+b$	
$(2A_{1,1}+3A)^{16}$	$[\alpha,\alpha]=a, [\beta,\beta]=b, [\gamma,\gamma]=a+b, [\alpha,\beta]=\kappa(a+b),$ $[\alpha,\gamma]=\lambda(a+b), [\beta,\gamma]=\mu(a+b)$	
$(2A_{1,1}+3A)^{17}$	$[\alpha,\alpha]=a, [\beta,\beta]=b, [\gamma,\gamma]=-(a+b), [\alpha,\beta]=\kappa(a+b),$ $[\alpha,\gamma]=\lambda(a+b), [\beta,\gamma]=\mu(a+b)$	
$(2A_{1,1}+3A)^{18}$	$[\alpha,\alpha]=a, [\beta,\beta]=-a, [\gamma,\gamma]=b, [\alpha,\beta]=\kappa b, [\alpha,\gamma]=b,$ $[\beta,\gamma]=\mu b$	
$(2A_{1,1}+3A)^{19}$	$[\alpha,\alpha]=a, [\beta,\beta]=-a, [\gamma,\gamma]=b, [\alpha,\beta]=b, [\beta,\gamma]=\mu b$	
$(2A_{1,1}+3A)^{20}$	$[\alpha,\alpha]=a, [\beta,\beta]=-a, [\gamma,\gamma]=b, [\beta,\gamma]=b$	
$(2A_{1,1}+3A)^{21}$	$[\alpha,\alpha]=a, [\beta,\beta]=-a, [\alpha,\beta]=b, [\alpha,\gamma]=b, [\beta,\gamma]=\mu b$	
$(2A_{1,1}+3A)^{22}$	$[\alpha,\beta]=a, [\beta,\gamma]=b, [\alpha,\gamma]=b$	
$(2A_{1,1}+3A)^{23}$	$[\alpha,\beta]=a, [\alpha,\gamma]=b$	

REFERENCES

Backhouse, N. B. (1977). *Proc. 6th Int. Colloq. Group Theoretical Methods in Physics*, Tubingen.

Backhouse, N. B. (1978a). *J. Math. Phys.* **19**, 2400–2402.

Backhouse, N. B. (1978b). *Lect. Notes Phys.* **79**, 364.

Balantekin, A. B. and Bars, I. (1982). *J. Math. Phys.* **23**, 1239–1247.

Balantekin, A. B. and Bars, I. (1984). *J. Phys. A* **17**, 1573–1578.

Beckers, J., Dehin, D., and Hussin, V. (1988a). *J. Math. Phys.* **29**, 1705–1711.

Beckers, J., Dehin, D., and Hussin, V. (1988b). *J. Phys. A* **21**, 651–667.

Berele, A. and Regev, A. (1987). *Adv. Math.* **64**, 118–175.

Brebner, M. A. and Grad, J. (1982). *Linear Algebr. Appl.* **43**, 99–118.

Bunse-Gerstner, A. (1984). *Linear Algebr. Appl.* **58**, 43–68.

Corwin, L., Neeman, Y., and Sternberg, S. (1975). *Rev. Mod. Phys.* **47**, 573–603.

Freund, P. G. O. and Kaplansky, I. (1976). *J. Math. Phys.* **17**, 228.

Garvey, S. D., Tisseur, F., Friswell, M. I., Penny, J. E. T., and Prells, U. (2003). *Int. J. Numer. Meth. Engng.* **57**, 1643–1660.

Hegazi, A. (1999a). *Int. J. Theor. Phys.* **38**, 1735–1739.

Hegazi, A. (1999b). *Int. J. Theor. Phys.* **68**, 2681–2693.

Kac, V. G. (1977). *Ad. Math.* **26**, 8–96.

Kac, V. G. (1978). *Lect. Notes Math.* **676**, 597.

Matiadou, N. L. and Fellouris, A. (2005). *Mathematica Balkanica* **19**, 143–154.

Mubarakzyanov, G. M. Izv. Vysshikh Uschebn. Zavedenii Mat., *On solvable Lie algebras*, No. 1, **32**, 114; *The classification of the real structure of five-dimensional Lie algebras*, No. 3, **34**, 99; *Classification of solvable Lie algebras of sixth order with a non nilpotent basis element*, No. 4, **35**, 104; (1963).

Neveu, A. and Schwarz, J. (1971). *Nucl. Phys. B* **31**, 86.

Patera, J., Sharp, R. T., Winterintz, P., and Zassenhaus, H. (1976). *J. Math. Phys.* **17**, 986–994.

Pais, A. and Rittenberg, V. (1975). *J. Math. Phys.* **16**, 2062–2073.

Palev, T. D. (1989). *J. Math. Phys.* **30**, 1433–1442.

Salam, A. and Strathdee, J. (1974). *Nucl. Phys. B* **79**, 477.

Scheunert, M., Nahm, W., and Rittenberg, V. (1977). *J. Math. Phys.* **18**, 146–154.

Van der Jeugt, J. (1987). *J. Phys. A* **20**, 809–824.

Wess, J. and Zumino, B. (1974). *Nucl. Phys. B* **70**, 39.